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Flows of thin streams with free boundaries

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A method is developed for determining any thin steady two-dimensional potential flow with free and/or rigid boundaries in the presence of gravity. The flow is divided into a number of parts and in each part the flow and its free boundaries are represented as asymptotic series in powers of the slenderness ratio of the stream. There are three basic flows, having two, one and no free boundaries and called jet flow, wall flow and channel flow, respectively. First the three expansions for these flows are found, extending results of Keller & Weitz (1952). They are called outer expansions to distinguish them from the inner expansions which apply near the ends of the stream or at the junction of two different types of flow. The inner and outer expansions must be matched at a junction to find how the emerging flow is related to the entering flow. This process can be continued to build up any complex flow involving thin streams. The method is illustrated in the case of a wall flow that leaves the wall to become a jet, which includes the case of a waterfall treated by Clarke (1965) in a similar way. In part 2 (to be published) other inner expansions are found and matched to outer expansions, providing the ingredients for the construction of the solutions of many flow problems.

1. Introduction

To treat flows of thin streams with free boundaries, Keller & Weitz (1952, 1957) represented the flows and the free boundaries as asymptotic series in powers of the slenderness ratio of the stream. This led to equations for the successive determination of the coefficients in the series, but it left various integration constants undetermined. The work of Clarke (1965), who treated a waterfall by matched asymptotic expansions, indicated that the constants could be found by matching these outer expansions to suitable inner expansions at the ends of the stream. We shall show that this can be done, and thus we shall develop a method for handling a large class of flows involving thin streams. Bentwich (1968), Clarke (1968) and Ackerberg (1968*a*, *b*) have treated special flows by matched expansions using the reciprocal of the Froude number as the small parameter.

For simplicity we consider only steady two-dimensional potential flows of thin streams. We call a part of the flow a jet if it has two free boundaries, a wall flow if it has one free and one rigid boundary, and a channel flow if it has two rigid boundaries. In § 2 we formulate the method and in §§ 3–5 we obtain the complete outer asymptotic expansion for each of these three types of flow, except for certain constants, thus extending the results of Keller & Weitz (1952). In § 6 we find the inner expansion for the junction of a wall flow and a jet flow, and in § 7 we match it to the outer expansions for wall and jet flows. In this way we obtain the three expansions for any wall flow which becomes a jet or vice versa. The results for this problem reduce to those of Clarke (1965) when specialized to his case.

There are two other basic inner expansions, one for the junction of a wall flow and a channel flow and the other for the junction of a channel flow and a jet. These and other inner expansions will be found in part 2 (to be published) and matched to the appropriate outer expansions. By using the three outer expansions and three or more inner expansions, it is possible to obtain the expansions for a great variety of flows involving thin streams. This can be done by analysing the flow as a sequence of parts, using the appropriate expansion for each part and matching it to the expansions of the adjoining parts. The possibility of analysing an elliptic problem in this way, as if it were hyperbolic, arises because the detailed influence of a junction decays exponentially with distance away from it divided by the slenderness ratio. Therefore it is negligible compared with the power series which our analysis includes.

The leading term of each inner expansion is not influenced by gravity, and generally not by the curvature of the rigid boundaries either. Therefore it can be found explicitly by the hodograph method and conformal mapping of the Schwarz–Christoffel type. In fact each known solution which has been found in this way is the leading term in a certain inner expansion, so it has a wide range of applicability, and may be viewed as a canonical solution for the junction or orifice it represents.

2. Formulation of the method

A steady two-dimensional potential flow can be described by giving the complex position $z' = x' + iy'$ as an analytic function of the complex potential $\phi' + i\psi'$, where ϕ' is the potential function and ψ' is the stream function. We suppose that the flow is bounded, at least in part, by the two streamlines $\psi' = 0$ and $\psi' = -Q$, where Q is the flux in the stream. Some portions of these streamlines will be required to lie on fixed boundary curves in the z' plane and other portions of these streamlines will be free. On a fixed boundary curve, the real and imaginary parts of z' must satisfy the equation of the curve. On a free portion of a streamline the constancy of the pressure and the Bernoulli equation yield $|dz'/d\phi'|^{-2} + 2gy' = U^2$, where U is a constant with the dimensions of velocity and g is the acceleration of gravity. The problem of determining a flow is that of finding the function $z'(\phi' + iy')$ satisfying a specified one of these conditions on each portion of the boundary, together with suitable conditions at $\phi' = \pm\infty$.

It is convenient to let L denote a typical length along the stream and to introduce $h = Q/U$, which denotes a typical width of the stream. Then $\epsilon = h/L$ is the slenderness ratio, which is a measure of the thinness of the stream. We also introduce the dimensionless quantities z, ϕ, ψ and γ defined by

$$z' = Lz, \quad \phi' = LU\phi, \quad \psi' = hU\psi, \quad \gamma = 2gL/U^2, \quad \epsilon = h/L. \tag{2.1}$$

The Froude number of the flow is γ^{-1}, which measures the relative importance of gravitational and inertial forces. In terms of these variables the boundary conditions become

$$|dz/d\phi|^2 = (1-\gamma \operatorname{Im} z)^{-1} \quad \text{on a free streamline,} \tag{2.2}$$

$$\operatorname{Im} z = \eta(\operatorname{Re} z) \quad \text{on a fixed boundary.} \tag{2.3}$$

Here $y = \eta(x)$ is the given equation of the fixed boundary.

The flow is now described by the analytic function $z(\phi + i\epsilon\psi, \epsilon)$, which depends explicitly on ϵ, and which is defined in the strip $0 \geqslant \psi \geqslant -1$ of the ϕ, ψ plane. We assume that each of the bounding streamlines is divided into a finite number of intervals in each of which either (2.2) or (2.3) holds, with a different function η_j given in each interval where (2.3) holds. Furthermore, we assume that each of these functions η_j is infinitely differentiable, so that if a given rigid boundary has discontinuities in its derivatives each portion between discontinuities is viewed as a separate interval. Let the combined set of end points ot the intervals on the two bounding streamlines be denoted by $-\infty = \phi_0, \phi_1, \ldots, \phi_{N+1} = +\infty$ with $\phi_j < \phi_{j+1}$, $j = 1, \ldots, N-1$. One of the finite ϕ_j can be chosen arbitrarily by choosing the origin of ϕ and the other $N-1$ must be determined. Therefore they will depend upon ϵ, so we shall write $\phi_j = \phi_j(\epsilon)$.

In each of the intervals into which the strip is divided by the points $\phi_j(\epsilon)$ just one of the boundary conditions (2.2) or (2.3) applies on each of the bounding streamlines. Thus there are three kinds of interval, which have respectively two, one or no free streamlines. If the length of the interval remains finite as ϵ tends to zero, we call these flows respectively jet, wall and channel flow. For each of them, we assume that, as ϵ tends to zero, z has an asymptotic power series expansion in ϵ which we write in the form

$$z(\phi + i\epsilon\psi, \epsilon) \sim \sum_{n=0}^{\infty} z_n(\phi + i\epsilon\psi)\,\epsilon^n, \quad \phi_j(\epsilon) < \phi < \phi_{j+1}(\epsilon). \tag{2.4}$$

The expansion coefficients z_n depend upon j, but we shall not indicate that explicitly.

We call the expansions (2.4) 'outer' expansions to distinguish them from the 'inner' expansions which hold in the neighbourhoods of the points $\phi_j(\epsilon)$ where two different flows join together. In the z plane the junctions occur at

$$z[\phi_j(\epsilon) + i\epsilon\psi, \epsilon].$$

To write the inner expansion near ϕ_j we introduce new variables ϕ'' and z'' defined by

$$\left.\begin{aligned} \phi'' &= \{\phi - \phi_j(0)\}/\epsilon = \{\phi' - \phi_j'(0)\}/hU, \\ z''(\phi'' + i\psi, \epsilon) &= \{z - z[\phi_j(0), 0]\}/\epsilon = \{z' - z'[\phi_j'(0), 0]\}/h. \end{aligned}\right\} \tag{2.5}$$

Here $\phi_j' = LU\phi_j$. In terms of these variables the flow is described by an analytic function $z''(\phi''+i\psi,\epsilon)$ defined in the strip $-1 \leqslant \psi \leqslant 0$ and the boundary conditions (2.2) and (2.3) become

$$|dz''/d\phi''|^2 = (1-\epsilon\gamma\,\mathrm{Im}\,z''-\gamma\,\mathrm{Im}\,z[\phi_j(0),0])^{-1} \text{ on a free streamline,} \tag{2.6}$$

$$\epsilon\,\mathrm{Im}\,z''+\mathrm{Im}\,z[\phi_j(0),0] = \eta(\epsilon\,\mathrm{Re}\,z''+\mathrm{Re}\,z[\phi_j(0),0]) \text{ on a fixed boundary.} \tag{2.7}$$

We assume that as ϵ tends to zero z'' has the asymptotic expansion

$$z''(\phi''+i\psi,\epsilon) \sim \sum_{n=0}^{\infty} z_n''(\phi''+i\psi)\,\epsilon^n. \tag{2.8}$$

We call this an 'inner' expansion.

The coefficients z_n in the three kinds of outer expansion will be determined in §§ 3–5. The coefficients z_n'' for one kind of inner expansion will be found in § 6, and for various other kinds of inner expansion they will be found in part 2. Upon matching the successive outer and inner expansions, the full expansion for any flow involving a thin stream can be found. This will be illustrated in § 7 for a wall flow which leaves the wall as a jet.

3. Outer expansion for a jet (flow with two free boundaries)

To determine the z_n in (2.4) for a jet, we first rewrite (2.4) by expanding $z_n(\phi+i\epsilon\psi)$ in powers of ϵ, and obtain

$$z(\phi+i\epsilon\psi,\epsilon) \sim \sum_{s=0}^{\infty} h_s(\phi,\psi)\,\epsilon^s. \tag{3.1}$$

Here h_s is defined by

$$h_s(\phi,\psi) = \sum_{n=0}^{s} \frac{z_n^{(s-n)}(\phi)\,(i\psi)^{s-n}}{(s-n)!}. \tag{3.2}$$

We now substitute (3.1) into (2.2) and use the binomial theorem to obtain, with $k_i > 0$,

$$\sum_{s=0}^{\infty} \epsilon^s \sum_{k=0}^{s} \frac{\partial h_{s-k}}{\partial\phi}\frac{\partial \bar{h}_k}{\partial\phi} \sim [1-\gamma\,\mathrm{Im}\,z_0(\phi)]^{-1} + \sum_{s=1}^{\infty}\epsilon^s\sum_{j=1}^{s}\gamma^j \times [1-\gamma\,\mathrm{Im}\,z_0(\phi)]^{-j-1} \sum_{k_1+\ldots+k_j=s}\prod_{i=1}^{j}\mathrm{Im}\,h_{k_i}. \tag{3.3}$$

Here and elsewhere an overbar indicates a complex conjugate. Upon equating coefficients of powers of ϵ in (3.3) we obtain an infinite set of equations for the $z_n = x_n+iy_n$. We can write these equations as follows, after eliminating the h_s by means of (3.2):

$$\sum_{k=0}^{s}\sum_{n=0}^{s-k}\sum_{m=0}^{k}(-1)^{k-m}(i\psi)^{s-n-m}z_n^{(s-k-n+1)}\bar{z}_m^{(k-m+1)}/(s-k-n)!\,(k-m)!$$
$$= (1-\gamma y_0)^{-1}\delta_{s0} + \sum_{j=1}^{s}\gamma^j(1-\gamma y_0)^{-j-1}\sum_{k_1+\ldots+k_j=s}\prod_{i=1}^{j}\left\{\sum_{r=0}^{[\frac{1}{2}(k_i-1)]}\frac{(-1)^r\psi^{2r+1}}{(2r+1)!}x_{k_i-2r-1}^{(2r+1)} + \sum_{r=0}^{[\frac{1}{2}k_i]}\frac{(-1)^r\psi^{2r}}{(2r)!}y_{k_i-2r}^{(2r)}\right\} \quad (s=0,1,\ldots). \tag{3.4}$$

The symbol $[\alpha]$ denotes the greatest integer not exceeding α.

On $\psi = 0$ equation (3.4) simplifies to

$$\sum_{k=0}^{s} (x'_{s-k}x'_k + y'_{s-k}y'_k) = (1-\gamma y_0)^{-1}\delta_{s0} + \sum_{j=1}^{s} \gamma^j (1-\gamma y_0)^{-j-1} \sum_{\substack{k_1+\ldots+k_j=s \\ (k_j>0)}} \prod_{i=1}^{j} y_{k_i} \quad (s = 0, 1, \ldots), \tag{3.5}$$

where a prime indicates differentiation with respect to ϕ. We can write (3.5) more explicitly in the following form:

$$x_0'^2 + y_0'^2 = (1-\gamma y_0)^{-1}, \tag{3.6}$$

$$2(x'_0x'_s + y'_0y'_s) - \gamma(1-\gamma y_0)^{-2} y_s = g_s \quad (s = 1, 2, \ldots). \tag{3.7}$$

Here g_s is defined by

$$g_s = \sum_{j=2}^{s} \gamma^j (1-\gamma y_0)^{-j-1} \sum_{k_1+\ldots+k_j=s} \prod_{i=1}^{j} y_{k_i} - \sum_{k=1}^{s-1} (x'_{s-k}x'_k + y'_{s-k}y'_k) \quad (s = 1, 2, \ldots). \tag{3.8}$$

We see from (3.8) that g_s does not involve either x_j or y_j with $j \geqslant s$. Explicitly, the first three g_s are given by

$$g_1 = 0, \tag{3.9}$$

$$g_2 = \gamma^2(1-\gamma y_0)^{-3} y_1^2 - x_1'^2 - y_1'^2, \tag{3.10}$$

$$g_3 = 2\gamma^2(1-\gamma y_0)^{-3} y_1 y_2 + \gamma^3 (1-\gamma y_0)^{-4} y_1^3 - 2(x'_2x'_1 + y'_2y'_1). \tag{3.11}$$

The line $\psi = -1$ is also a free streamline on which (3.4) holds. By subtracting (3.5) from (3.4) and setting $\psi = -1$ we obtain in this case

$$x'_0y''_0 - y'_0x''_0 + \tfrac{1}{2}\gamma(1-\gamma y_0)^{-2} x'_0 = 0, \tag{3.12}$$

$$x'_0y''_s - y'_0x''_s + \tfrac{1}{2}\gamma(1-\gamma y_0)^{-2}x'_s + y''_0x'_s - x''_0y'_s + \gamma^2(1-\gamma y_0)^{-3}x'_0y_s = f_s \quad (s = 1, 2, \ldots). \tag{3.13}$$

In (3.13) f_s is defined by

$$\begin{aligned}2f_{s-1} = &- \sum_{\substack{k=0 \\ (n,m)\neq(s,s-1) \\ n+m\neq s}}^{s} \sum_{n=0}^{s-k} \sum_{m=0}^{k} (-1)^{k-m}(-i)^{s-n-m} z_n^{(s-k-n+1)} \bar{z}_m^{(k-m+1)} / (s-k-n)!\,(k-m)! \\ &+ \sum_{j=1}^{s} \gamma^j (1-\gamma y_0)^{-j-1} \sum_{\substack{k_1+\ldots+k_j=s \\ (k_i>0)}} \left\{ \prod_{i=1}^{j} \left(\sum_{r=0}^{[\frac{1}{2}(k_i-1)]} \frac{(-1)^{r+1}}{(2r+1)!} x^{(2r+1)}_{k_i-2r-1} + \sum_{r=0}^{[\frac{1}{2}k_i]} \frac{(-1)^r}{(2r)!} y^{(2r)}_{k_i-2r} \right) \right. \\ &\left. - \prod_{i=1}^{j} y_{k_i} \right\} + \gamma(1-\gamma y_0)^{-2} x'_{s-1} + 2\gamma^2(1-\gamma y_0)^{-3} x'_0 y_{s-1} \quad (s-1 = 1, 2, \ldots).\end{aligned} \tag{3.14}$$

Neither x_j nor y_j with $j \geqslant s$ occur in f_s. Explicitly, the first two f_s are given by

$$f_1 = \tfrac{1}{4}\gamma^2(1-\gamma y_0)^{-4}, \tag{3.15}$$

$$\begin{aligned}2f_2 = {} & x'_0x'''_1 + y'_0y'''_1 - \tfrac{1}{3}x_0^{\mathrm{iv}}y'_0 + \tfrac{1}{3}x'_0y_0^{\mathrm{iv}} - y'''_0x''_0 + x'''_0y''_0 + x'''_0x'_1 + y'''_0y'_1 \\ & - 2(x''_1x''_0 + y''_1y''_0 + y''_1x'_1 - x''_1y'_1) + \gamma(1-\gamma y_0)^{-2}(\tfrac{1}{6}x'''_0 - \tfrac{1}{2}y''_1) + 2\gamma^2(1-\gamma y_0)^{-3} \\ & \times (x'_0x'_1 + \tfrac{1}{2}y''_0x'_0 - x'_1y_1 - \tfrac{1}{2}y_1y''_0) + \gamma^3(1-\gamma y_0)^{-4} \\ & \times (-[x'_0]^3 + 3[x'_0]^2y_1 - 3x'_0[y_1]^2).\end{aligned} \tag{3.16}$$

For the jet flow, (3.6), (3.7), (3.12) and (3.13) all apply. The leading coefficients x_0 and y_0 satisfy the nonlinear first- and second-order ordinary differential equations (3.6) and (3.12), in which ϕ is the independent variable. For $s \geqslant 1$ the coefficients x_s and y_s satisfy the linear first- and second-order ordinary differential equations (3.7) and (3.13). These equations can be solved recursively starting with $s = 1$ because the right sides involve only the x_j and y_j with $j < s$. Thus apart from constants of integration, equations (3.6), (3.7) (3.12) and (3.13) determine the coefficients $z_n = x_n + iy_n$ in the outer expansion (3.1) for a jet.

Let us consider the solution $x_0(\phi)$, $y_0(\phi)$ of (3.6) and (3.12) which satisfies the initial conditions

$$x_0(0) = 0, \quad y_0(0) = a < \gamma^{-1}, \quad dy_0(0)/dx_0 = \tan\beta. \tag{3.17}$$

If $\beta \neq \pm\frac{1}{2}\pi$, the solution is given by the equations

$$y_0 = a + x_0 \tan\beta - bx_0^2, \tag{3.18}$$

$$\phi = 2(b/\gamma)^{\frac{1}{2}}[(1-\gamma a)\,x_0 - \tfrac{1}{2}\gamma x_0^2 \tan\beta + \tfrac{1}{3}\gamma b x_0^3]. \tag{3.19}$$

Here b is defined by

$$b = \gamma \sec^2\beta/4(1-\gamma a). \tag{3.20}$$

From (3.18) we see that the curve $x_0(\phi)$, $y_0(\phi)$ is a parabola, as we expect, unless $\gamma = 0$, in which case it is a straight line. If $\beta = \pm\frac{1}{2}\pi$ and $\gamma \neq 0$ the solution is

$$x_0 = 0, \tag{3.21}$$

$$y_0 = \gamma^{-1} - \gamma^{-1}[\mp\tfrac{3}{2}\gamma\phi + (1-\gamma a)^{\frac{3}{2}}]^{\frac{2}{3}}. \tag{3.22}$$

If $\beta = \pm\frac{1}{2}\pi$ and $\gamma = 0$ then $x_0 = 0$ and $y_0 = a \pm \phi$.

To find $x_s(\phi)$ and $y_s(\phi)$ for $s \geqslant 1$ in the case $\beta \neq \pm\frac{1}{2}\pi$, we note that (3.19) yields $x_0'(\phi) \neq 0$. Therefore we can solve (3.7) for x_s' to obtain

$$x_s' = \frac{g_s}{2x_0'} + \frac{\gamma(1-\gamma y_0)^{-2} y_s}{2x_0'} - \frac{y_0' y_s'}{x_0'}. \tag{3.23}$$

By using (3.18) and (3.6) we can rewrite (3.23) as

$$x_s' = g_s/2x_0' + 2b(x_0 y_s)' - y_s' \tan\beta. \tag{3.24}$$

Integration of (3.24) yields

$$x_s(\phi) = c_s + 2bx_0 y_s + (a_s - y_s)\tan\beta + \int_0^\phi \frac{g_s}{2x_0'}\,d\phi \quad (s = 1, 2, \ldots). \tag{3.25}$$

Here $c_s = x_s(0)$ and $a_s = y_s(0)$. Now we use (3.24) to eliminate x_s from (3.13) and obtain

$$y_s'' - \frac{2\gamma y_0'}{1-\gamma y_0} y_s' + \frac{\gamma^3}{4b}(1-\gamma y_0)^{-4} y_s = F_s. \tag{3.26}$$

The quantity F_s is defined by

$$F_s = \frac{1}{2}\left(\frac{\gamma}{b}\right)^{\frac{1}{2}}\left[f_s - \frac{y_0'' g_s}{2x_0'} + \frac{y_0'}{2}\left(\frac{g_s}{x_0'}\right)' - \frac{\gamma g_s}{4x_0'(1-\gamma y_0)^2}\right]. \tag{3.27}$$

The general solution of the homogeneous form of (3.26) is, with A_s and B_s arbitrary constants,

$$y_s = (1-\gamma y_0)^{-1}\left[A_s\left(x_0-\frac{\tan\beta}{2b}\right)+B_s\left(y_0+\frac{1}{2b}-\frac{1}{\gamma}\right)\right]. \qquad (3.28)$$

Thus the general solution of (3.26) is

$$y_s(\phi) = (1-\gamma y_0)^{-1}\left(x_0-\frac{\tan\beta}{2b}\right)\left[A_s+2(b\gamma)^{\frac{1}{2}}\int_0^{\phi}\left(y_0+\frac{1}{2b}-\frac{1}{\gamma}\right)(1-\gamma y_0)\,F_s\,d\phi'\right]$$
$$+(1-\gamma y_0)^{-1}\left(y_0+\frac{1}{2b}-\frac{1}{\gamma}\right)\left[B_s-2(b\gamma)^{\frac{1}{2}}\int_0^{\phi}\left(x_0-\frac{\tan\beta}{2b}\right)(1-\gamma y_0)\,F_s\,d\phi'\right]$$
$$(s = 1, 2, \ldots). \qquad (3.29)$$

In the special case $\gamma = 0$, equation (3.29) can be simplified to

$$y_s(\phi) = A_s+B_s\phi+\int_0^{\phi}(\phi-\phi')\,F_s(\phi')\,d\phi' \quad (s = 1, 2, \ldots), \quad (\gamma = 0). \qquad (3.30)$$

By using (3.29) or (3.30) in (3.25) we obtain $x_s(\phi)$.

For $s = 1$, equations (3.12), (3.15) and (3.27) yield

$$F_1 = \tfrac{1}{8}\gamma^2(\gamma/b)^{\frac{1}{2}}\,(1-\gamma y_0)^{-4}. \qquad (3.31)$$

Then (3.29) leads to the following result, which also follows directly from (3.26):

$$y_1(\phi) = A_1(1-\gamma y_0)^{-1}\left(x_0-\frac{\tan\beta}{2b}\right)+B_1(1-\gamma y_0)^{-1}\left(y_0+\frac{1}{2b}-\frac{1}{\gamma}\right)+\frac{1}{2}\left(\frac{b}{\gamma}\right)^{\frac{1}{2}}. \qquad (3.32)$$

Now (3.25) yields

$$x_1(\phi) = x_1(0)+2bx_0(\phi)\,y_1(\phi)+[y_1(0)-y_1(\phi)]\tan\beta. \qquad (3.33)$$

Further x_s and y_s can be found in the same way. In the special case $\gamma = 0$ it follows by induction that $f_s = 0$ and $g_s =$ constant for each s. As a consequence (3.25) and (3.30) show that x_s and y_s are linear functions of ϕ for each s.

Later we shall need A_1 and B_1 in terms of $y_1(0)$ and $y_1'(0)$. By setting $\phi = 0$ in (3.32) and in the differentiated form of (3.32) and solving the resulting equations, we obtain

$$A_1 = \sec\beta(1-\gamma a)^{\frac{3}{2}}\,(1-2\sin^2\beta)\,y_1'(0)-2\gamma\cos\beta\sin\beta$$
$$\times[y_1(0)-\sec\beta/4(1-\gamma a)^{\frac{1}{2}}], \qquad (3.34)$$

$$B_1 = 2\sin\beta(1-\gamma a)^{\frac{3}{2}}\,y_1'(0)+\gamma(1-2\sin^2\beta)\,[y_1(0)-\sec\beta/4(1-\gamma a)^{\frac{1}{2}}]. \qquad (3.35)$$

When $\beta = \pm\frac{1}{2}\pi$, equation (3.21) yields $x_0' = 0$ so (3.7) and (3.13) become

$$y_s' \mp \tfrac{1}{2}\gamma(1-\gamma y_0)^{-\frac{3}{2}}\,y_s = \pm\tfrac{1}{2}(1-\gamma y_0)^{\frac{1}{2}}\,g_s, \qquad (3.36)$$

$$x_s'' \mp \gamma(1-\gamma y_0)^{-\frac{3}{2}}\,x_s' = \mp(1-\gamma y_0)^{\frac{1}{2}}f_s. \qquad (3.37)$$

The solutions of (3.36) for which $y_s(0) = a_s$, and of (3.37) for which $x_s(0) = c_s$ and $x_s'(0) = b_s$, are

$$y_s(\phi) = a_s(1-\gamma a)^{\frac{1}{2}}\,[(1-\gamma a)^{\frac{3}{2}}\mp\tfrac{3}{2}\gamma\phi]^{-\frac{1}{3}}$$
$$\pm\tfrac{1}{2}[(1-\gamma a)^{\frac{3}{2}}\mp\tfrac{3}{2}\gamma\phi]^{-\frac{1}{3}}\int_0^{\phi}[(1-\gamma a)^{\frac{3}{2}}\mp\tfrac{3}{2}\gamma\phi']^{\frac{2}{3}}\,g_s(\phi')\,d\phi', \qquad (3.38)$$

$$x_s(\phi) = c_s \pm 2b_s(1-\gamma a)\gamma^{-1}[(1-\gamma a)^{\frac{1}{2}}-(1-\gamma y_0)^{\frac{1}{2}}] \mp \int_0^{\phi}[1-\gamma y_0(\phi')]^{-1}\int_0^{\phi'}[1-\gamma y_0(\phi'')]^{\frac{3}{2}} f_s(\phi'')\, d\phi''\, d\phi'. \quad (3.39)$$

For $s = 1$, equation (3.9) shows that $g_1 = 0$ and (3.18) gives f_1. Then (3.38) and (3.39) become

$$y_1(\phi) = a_1(1-\gamma a)^{-\frac{1}{2}}[(1-\gamma a)^{\frac{3}{2}} \pm \tfrac{3}{2}\gamma\phi]^{-\frac{1}{3}}, \quad (3.40)$$

$$x_1(\phi) = c_1 \pm 2b_1(1-\gamma a)\gamma^{-1}[(1-\gamma a)^{\frac{1}{2}}-(1-\gamma y_0)^{\frac{1}{2}}] \mp \tfrac{1}{2}(1-\gamma y_0)^{-\frac{1}{2}} \mp \tfrac{1}{2}(1-\gamma y_0)^{\frac{1}{2}}(1-\gamma a)^{-1} \pm (1-\gamma a)^{-\frac{1}{2}}. \quad (3.41)$$

4. Outer expansion for a wall flow (one free and one fixed boundary)

We next treat a wall flow with a free boundary $\psi = 0$ and a fixed boundary $y = \eta(x)$ on which $\psi = -1$. The results of § 3 up to (3.11) apply in the present case, since they involve only the free boundary $\psi = 0$. For the fixed boundary we have

$$\operatorname{Im} z = \eta(\operatorname{Re} z) \quad \text{on} \quad \psi = -1. \quad (4.1)$$

By using (3.1) in (4.1) we obtain

$$\sum_{s=0}^{\infty} \epsilon^s \operatorname{Im} h_s(\phi, -1) \sim \eta\left[\sum_{k=0}^{\infty} \epsilon^k \operatorname{Re} h_k(\phi, -1)\right]. \quad (4.2)$$

We now expand the right side of (4.2) in powers of ϵ and note that $\operatorname{Re} h_0 = x_0(\phi)$. Then (4.2) becomes, with $k_i > 0$,

$$\sum_{s=0}^{\infty} \epsilon^s \operatorname{Im} h_s(\phi, -1) \sim \eta(x_0) + \sum_{s=1}^{\infty} \epsilon^s \sum_{j=1}^{\infty} \frac{\eta^{(j)}(x_0)}{j!} \times \sum_{k_1+\ldots+k_j=s} \operatorname{Re} h_{k_1}(\phi, -1)\ldots\operatorname{Re} h_{k_j}(\phi, -1). \quad (4.3)$$

Equating coefficients of ϵ^s in (4.3) yields

$$\operatorname{Im} h_s(\phi, -1) = \delta_{s0}\eta(x_0) + \sum_{j=1}^{s} \frac{\eta^{(j)}(x_0)}{j!} \sum_{k_1+\ldots+k_j=s} \operatorname{Re} h_{k_1}(\phi, -1)\ldots\operatorname{Re} h_{k_j}(\phi, -1) \quad (s = 0, 1, \ldots). \quad (4.4)$$

From (3.2) we find that

$$\operatorname{Re} h_s(\phi, -1) = \sum_{j=0}^{[\frac{1}{2}s]} \frac{(-1)^j}{(2j)!} x_{s-2j}^{(2j)} + \sum_{j=0}^{[\frac{1}{2}(s-1)]} \frac{(-1)^j}{(2j+1)!} y_{s-2j-1}^{(2j+1)}, \quad (4.5)$$

$$\operatorname{Im} h_s(\phi, -1) = -\sum_{j=0}^{[\frac{1}{2}(s-1)]} \frac{(-1)^j}{(2j+1)!} x_{s-2j-1}^{(2j+1)} + \sum_{j=0}^{[\frac{1}{2}s]} \frac{(-1)^j}{(2j)!} y_{s-2j}^{(2j)}. \quad (4.6)$$

By using (4.5) and (4.6) in (4.4), we obtain a system of equations involving $x_s(\phi)$ and $y_s(\phi)$. These equations together with (3.6) and (3.7) can be used for the successive determination of the x_s and y_s.

For $s = 0$, equation (4.4) becomes

$$y_0 = \eta(x_0). \quad (4.7)$$

Upon using (4.7) in (3.6), we find that $x_0(\phi)$ must satisfy the first-order ordinary differential equation

$$x_0'(\phi) = [\{1+(\eta'(x_0))^2\}\{1-\gamma\eta(x_0)\}]^{-\frac{1}{2}}. \tag{4.8}$$

When $x_0(\phi)$ is found from (4.8), then (4.7) yields $y_0(\phi)$.

For $s \geqslant 1$, equation (4.4) can be written as follows:

$$y_s = \eta'(x_0)\,x_s + E_s. \tag{4.9}$$

Here E_s is defined by

$$\begin{aligned} E_s = \sum_{j=0}^{[\frac{1}{2}(s-1)]} \frac{(-1)^j}{(2j+1)!} x_{s-2j-1}^{(2j+1)} - \sum_{j=1}^{[\frac{1}{2}s]} \frac{(-1)^j}{(2j)!} y_{s-2j}^{(2j)} + \sum_{j=1}^{s} \frac{\eta^{(j)}(x_0)}{j!} \\ \times \sum_{k_1+\dots+k_j=s} \operatorname{Re} h_{k_1}(\phi,-1)\dots\operatorname{Re} h_{k_j}(\phi,-1) - \eta'(x_0)\,x_s. \end{aligned} \tag{4.10}$$

It is important to observe that E_s does not involve x_j or y_j with $j \geqslant s$, the terms in x_s and y_s having been written explicitly in (4.9). We now use (4.9) to eliminate y_s and y_s' from (3.7) and (4.7) to eliminate y_0. Then (3.7) becomes

$$x_s' + \alpha x_s = \beta_s. \tag{4.11}$$

Here α and β_s are defined by

$$\alpha = \frac{-\gamma(1-\gamma y_0)^{-2}\eta'(x_0) + 2y_0'\eta''(x_0)\,x_0'}{2x_0'\{1+[\eta'(x_0)]^2\}} = -[\log x_0'(\phi)]' \tag{4.12}$$

and

$$\beta_s = \frac{g_s + \gamma(1-\gamma y_0)^{-2}E_s - 2y_0'E_s'}{2x_0'\{1+[\eta'(x_0)]^2\}}. \tag{4.13}$$

The solution of (4.11) is

$$\begin{aligned} x_s(\phi) &= \exp\left[-\int_{\phi_0}^{\phi} \alpha(\phi')\,d\phi'\right]\left\{x_s(\phi_0) + \int_{\phi_0}^{\phi} \exp\left[\int_{\phi_0}^{t} \alpha(\phi')\,d\phi'\right]\beta_s(t)\,dt\right\} \\ &= x_0'(\phi)\left\{\frac{x_s(\phi_0)}{x_0'(\phi_0)} + \int_{\phi_0}^{\phi} \frac{\beta_s(t)}{x_0'(t)}\,dt\right\}, \quad s = 1,2,\dots. \end{aligned} \tag{4.14}$$

Then $y_s(\phi)$ is given by (4.9).

For $s = 1$ we obtain from (4.14) and (4.9), after some calculation,

$$\begin{aligned} x_1(\phi) = x_1(\phi_0)&\left\{\frac{[1-\gamma\eta(x_0(\phi_0))]\{1+[\eta'(x_0(\phi_0))]^2\}}{[1-\gamma\eta(x_0(\phi))]\{1+[\eta'(x_0(\phi))]^2\}}\right\}^{\frac{1}{2}} - [1-\gamma\eta(x_0(\phi))]^{-\frac{1}{2}} \\ &\times\{1+[\eta'(x_0(\phi))]^2\}^{-\frac{1}{2}}\left\{\eta'[x_0(\phi)] - \eta'[x_0(\phi_0)] - \tan^{-1}\eta'[x_0(\phi)]\right. \\ &\left. + \tan^{-1}\eta'[x_0(\phi_0)] - \frac{\gamma}{2}\int_{x_0(\phi_0)}^{x_0(\phi)} [1-\eta(t)]^{-1}\,dt\right\}, \end{aligned} \tag{4.15}$$

$$y_1(\phi) = \left\{\frac{1+\{\eta'[x_0(\phi)]\}^2}{1-\gamma\eta[x_0(\phi)]}\right\}^{\frac{1}{2}} + \eta'[x_0(\phi)]\,x_1(\phi). \tag{4.16}$$

5. Outer expansion for a channel flow (two fixed boundaries)

Finally we shall determine the outer expansion for a channel flow bounded by the two fixed boundaries $y = \eta(x)$, on which $\psi = -1$, and $y = \eta(x) + \epsilon\zeta(x)$, on which $\psi = 0$. The formulation of § 3 applies to this case up to (3.2) and the results of § 4 apply up to (4.10), except for (4.8). We now impose the condition

$$\operatorname{Im} z = \eta(\operatorname{Re} z) + \epsilon\zeta(\operatorname{Re} z) \quad \text{on} \quad \psi = 0. \tag{5.1}$$

Then as in § 4, we use (3.1) in (5.1), expand the right side in powers of ϵ and equate coefficients of like powers of ϵ to obtain

$$\begin{aligned}\operatorname{Im} h_s(\phi, 0) = \delta_{s0}\eta(x_0) + \sum_{j=1}^{s} \frac{\eta^{(j)}(x_0)}{j!} \sum_{k_1+\ldots+k_j=s} \operatorname{Re} h_{k_1}(\phi, 0)\ldots\operatorname{Re} h_{k_j}(\phi, 0) \\ + \delta_{s1}\zeta(x_0) + \sum_{j=1}^{s-1} \frac{\zeta^{(j)}(x_0)}{j!} \sum_{k_1+\ldots+k_j=s-1} \operatorname{Re} h_{k_1}(\phi, 0)\ldots\operatorname{Re} h_{k_j}(\phi, 0) \quad (s = 0, 1, \ldots).\end{aligned} \tag{5.2}$$

From (3.2) we obtain

$$\operatorname{Re} h_s(\phi, 0) = \operatorname{Re} z_s(\phi) = x_s(\phi), \tag{5.3}$$

$$\operatorname{Im} h_s(\phi, 0) = \operatorname{Im} z_s(\phi) = y_s(\phi). \tag{5.4}$$

For $s = 0$ equations (5.2) and (5.4) yield $y_0 = \eta(x_0)$, which is just the result (4.7) obtained from (4.4) with $s = 0$. Thus both boundary conditions yield (4.7) for $s = 0$. For $s \geqslant 1$, equation (5.2) becomes, when (5.3) and (5.4) are used in it,

$$y_s - \eta'(x_0)\,x_s - (1-\delta_{s1})\,[\tfrac{1}{2}\eta''(x_0)\,x_1(2-\delta_{s2}) + \zeta'(x_0)]\,x_{s-1} - \delta_{s1}\zeta(x_0) = G_{s-1} \quad (s = 1, 2, \ldots). \tag{5.5}$$

Here G_{s-1} is defined by

$$\begin{aligned}G_{s-1} = +\frac{\eta''(x_0)}{2}\sum_{k=2}^{s-2} x_k x_{s-k} + \sum_{j=3}^{s} \frac{\eta^{(j)}(x_0)}{j!} \sum_{k_1+\ldots+k_j=s} x_{k_1}\ldots x_{k_j} \\ + \sum_{j=2}^{s-1} \frac{\zeta^{(j)}(x_0)}{j!} \sum_{k_1+\ldots+k_j=s-1} x_{k_1}\ldots x_{k_j} \quad (s = 1, 2, \ldots).\end{aligned} \tag{5.6}$$

We note that G_{s-1} involves only x_j and y_j with $j \leqslant s-2$ and that $G_0 = G_1 = 0$.

Equations (5.5) and (4.9) can be used to determine the x_s and y_s successively. To determine them we should subtract (5.5) from (4.9), but we observe that x_s and y_s cancel. Therefore we first write out the terms in x_{s-1} and y_{s-1} in (4.9) explicitly and obtain

$$y_s - \eta'(x_0)\,x_s - x'_{s-1} - \eta'(x_0)\,y'_{s-1} - \tfrac{1}{2}(1-\delta_{s1})\,\eta''(x_0)\,[(2-\delta_{s2})\,x_1 + 2y'_0]x_{s-1} = H_{s-1}, \tag{5.7}$$

where H_{s-1}, which involves only x_j and y_j with $j \leqslant s-2$, is given by

$$H_{s-1} = E_s - x'_{s-1} - \eta'(x_0)\,y'_{s-1} - \tfrac{1}{2}(1-\delta_{s1})\,\eta''(x_0)\,[(2-\delta_{s2})\,x_1 + 2y'_0]\,x_{s-1}. \tag{5.8}$$

Now we subtract (5.5) from (5.7) and replace s by $s+1$ to obtain

$$\eta'(x_0)\,y'_s + x'_s + (1-\delta_{s0})\,[\eta''(x_0)\,y'_0 - \zeta'(x_0)]\,x_s - \delta_{s0}\zeta(x_0) = G_s - H_s \quad (s = 0, 1, \ldots). \tag{5.9}$$

Equations (5.9) and (5.5) can be used to determine the x_s and y_s.

For $s = 0$, equation (5.9) becomes

$$\eta'(x_0)\, y_0' + x_0' - \zeta(x_0) = 0. \tag{5.10}$$

By using (4.7) for y_0 in (5.10), we obtain

$$x_0' = \zeta(x_0)\{1 + [\eta'(x_0)]^2\}^{-1}. \tag{5.11}$$

For $s \geqslant 1$ we use (4.7) for y_0 and (5.7) for y_s in (5.9) with the result

$$\begin{aligned} x_s' + \sigma x_s = \{1 + [\eta'(x_0)]^2\}^{-1}\{G_s - H_s - \eta'(x_0)\, H_{s-1}' - \eta'(x_0) \\ \times \{\eta'(x_0)\, y_{s-1}'' + \tfrac{1}{2}(1-\delta_{s1})\, \eta''(x_0)\, [(2-\delta_{s2})\, x_1 + 2y_0']\, x_{s-1} + x_{s-1}'\}'\}. \end{aligned} \tag{5.12}$$

Here σ is defined by

$$\sigma = \{1 + [\eta'(x_0)]^2\}^{-1}\{2\eta'\eta'' x_0' - \zeta'\} = \left[\log \frac{\{1 + [\eta'(x_0)]^2\}}{\zeta(x_0)}\right]'. \tag{5.13}$$

Solving (5.12) for x_s yields

$$\begin{aligned} x_s(\phi) = \exp\left[-\int_{\phi_0}^{\phi} \sigma(t)\, dt\right]\left\{x_s(\phi_0) + \int_{\phi_0}^{\phi} \exp\left[\int_{\phi_0}^{\tau} \sigma(t)\, dt\right]\{1 + [\eta'(x_0)]^2\}^{-1}\right. \\ \left. \times [G_s - H_s - \eta' H_{s-1}' - \eta'\{\eta' y_{s-1}' + \tfrac{1}{2}(1-\delta_{s1})\, \eta''\, [(2-\delta_{s2})\, x_1 + 2y_0']\, x_{s-1} + x_{s-1}'\}']\, d\tau\right\} \\ (s = 1, 2, \ldots). \end{aligned} \tag{5.14}$$

Then $y_s(\phi)$ for $s \geqslant 1$ is determined by (5.7), which completes the determination of the outer expansion in this case, except for the constants of integration.

For $s = 1$ we can evaluate (5.14) for $x_1(\phi)$ by using (4.10) for E_s in (5.8) to obtain H_s and (5.6) for G_s. After some calculation we obtain

$$\begin{aligned} x_1(\phi) = \frac{\zeta[x_0(\phi)]}{1 + \{\eta'[x_0(\phi)]\}^2}\left[\frac{1 + \{\eta'[x_0(\phi_0)]\}^2}{\zeta[x_0(\phi_0)]}\, x_1(\phi_0) - \frac{\eta'[x_0(\phi)]}{2} + \frac{\eta'[x_0(\phi_0)]}{2}\right. \\ \left. - \int_{x_0(\phi_0)}^{x_0(\phi)} \frac{\eta'(x)\, \zeta'(x)}{\zeta(x)}\, dx\right]. \end{aligned} \tag{5.15}$$

Then (5.7) and (5.15) yield

$$y_1(\phi) = \zeta[x_0(\phi)] + \eta'[x_0(\phi)]\, x_1(\phi). \tag{5.16}$$

6. Inner expansion for a wall flow which becomes a jet

We have now found the three kinds of outer expansions. We shall next find the inner expansion for a wall flow which leaves the wall and becomes a jet. We choose the origin of ϕ at the point where the flow leaves the wall, so that $\phi_1 = 0$, and we also choose this point as the origin in the z plane, so that $\eta(0) = 0$. Therefore we have a wall flow for $\phi < 0$. To find the coefficients z_n'' in the inner expansion (2.8) we must use (2.6) and (2.7). In the present case these conditions become

$$\left.\begin{aligned} |dz''/d\phi''|^2 = (1 - \epsilon\gamma \operatorname{Im} z'')^{-1} \quad \text{on} \quad \psi = 0, \quad -\infty < \phi'' < +\infty \\ \text{and on} \qquad \psi = -1, \quad \phi'' > 0, \end{aligned}\right\} \tag{6.1}$$

$$\epsilon y'' = \eta(\epsilon x''), \quad x'' \leqslant 0 \quad \text{on} \quad \psi = -1, \quad \phi'' < 0. \tag{6.2}$$

From now on we shall omit the primes from z'' and ϕ'', and use primes to denote derivatives.

We shall seek an expansion for z of the form

$$z(\phi+i\psi,\epsilon) \sim \sum_{n=0}^{\infty} z_n(\phi+i\psi)\,\epsilon^n. \tag{6.3}$$

Upon using (6.3) in (6.1) and expanding both sides we obtain on the free streamlines, with $n_i \geqslant 0$,

$$\sum_{k=0}^{\infty} \epsilon^k \sum_{j=0}^{k} (x'_{k-j}x'_j + y'_{k-j}y'_j) \sim 1 + \sum_{k=1}^{\infty} \epsilon^k \sum_{j=1}^{k} \gamma^j \sum_{n_1+\ldots+n_j=k-j} y_{n_1}\cdots y_{n}. \tag{6.4}$$

Equating coefficients of ϵ^0 in (6.4) yields, on free streamlines,

$$x_0'^2 + y_0'^2 = 1. \tag{6.5}$$

The equation obtained by equating coefficients of ϵ^k in (6.4), valid on free streamlines, can be written in the form

$$x'_0x'_k + y'_0y'_k = J_k \equiv \frac{1}{2}\sum_{j=1}^{k} \gamma^j \sum_{n_1+\ldots+n_j=k-j} y_{n_1}\cdots y_{n_j} - \frac{1}{2}\sum_{j=1}^{k-1}(x'_{k-j}x'_j + y'_{k-j}y'_j) \quad (k = 1,2,\ldots). \tag{6.6}$$

On the fixed boundary (6.2) and (6.3) yield

$$\epsilon \sum_{n=0}^{\infty} \epsilon^n y_n \sim \sum_{j=1}^{\infty} \frac{\eta^{(j)}(0)}{j!}\left(\epsilon \sum_{n=0}^{\infty} \epsilon^n x_n\right)^j. \tag{6.7}$$

Equating coefficients of ϵ^{k+1} in (6.7) yields on the fixed boundary $\psi = -1$, $\phi < 0$

$$y_k - \eta'(0)\,x_k = K_k \equiv \sum_{j=2}^{k+1} \frac{\eta^{(j)}(0)}{j!} \sum_{n_1+\ldots+n_j=k+1-j} x_{n_1}\cdots x_{n_j} \quad (k = 0,1,\ldots). \tag{6.8}$$

The right sides of (6.6) and (6.8), denoted J_k and K_k respectively, do not involve x_s or y_s with $s \geqslant k$, so these equations can be used to find the x_k and y_k successively. Each $z_k(\phi+i\psi)$ must be analytic in the strip $-1 \leqslant \psi \leqslant 0$ and must satisfy (6.5) if $k = 0$ or (6.6) if $k \neq 0$ on the free streamlines and (6.8) on the fixed boundary. In addition $z_k(-i) = 0$ in order that the origin be at $\phi = 0$, $\psi = -1$.

To find z_0 we set $k = 0$ in (6.8) and obtain

$$y_0 - \eta'(0)\,x_0 = 0 \quad \text{on} \quad \psi = -1,\ \phi < 0. \tag{6.9}$$

Both (6.5) and (6.9) are satisfied identically by the linear function

$$z_0(\phi+i\psi) = e^{i\theta}(\phi+i\psi+i), \tag{6.10}$$

where

$$\theta = \tan^{-1}\eta'(0). \tag{6.11}$$

In addition $z_0 = 0$ at $\phi = 0$, $\psi = -1$ as we require, so we shall choose (6.10) for z_0.

To find z_k, $k \geqslant 1$, it is convenient to introduce w_k defined by

$$w_k = e^{-i\theta} z_k. \tag{6.12}$$

Then (6.6) and (6.8) can be written, on free and fixed streamlines respectively, as

$$\operatorname{Re} w'_k = J_k, \quad k = 1,2,\ldots, \quad \text{on} \quad \psi = 0, -1,\ \phi > 0. \tag{6.13}$$

$$\operatorname{Im} w_k = K_k \cos\theta, \quad k = 1,2,\ldots, \quad \text{on} \quad \psi = -1,\ \phi < 0. \tag{6.14}$$

Differentiation of (6.14) with respect to ϕ yields

$$\operatorname{Im} w_k' = K_k' \cos\theta, \quad k = 1, 2, \ldots, \quad \text{on} \quad \psi = -1, \phi < 0. \tag{6.15}$$

The problem of finding a function w_k' analytic in the strip $-1 \leqslant \psi \leqslant 0$, satisfying (6.13) and (6.15) on the boundaries, is a straightforward one. The solution is

$$w_k'(\phi + i\psi) = \frac{2i(1+e^{-\pi(\phi+i\psi)})^{\frac{1}{2}}}{\pi} \int_0^\infty \left\{ \frac{p_k(\sigma)}{1+\sigma^2+e^{-\pi(\phi+i\psi)}} + \frac{q_k(\sigma)}{1-\sigma^2+e^{-\pi(\phi+i\psi)}} \right\} d\sigma. \tag{6.16}$$

Here $p_k(\sigma)$ and $q_k(\sigma)$ are defined by

$$p_k(\sigma) = \cos\theta K_k'[-\pi^{-1}\log(1+\sigma^2), -1], \tag{6.17}$$

$$q_k(\sigma) = \begin{cases} J_k[-\pi^{-1}\log(1-\sigma^2), -1] & \text{for} \quad 0 \leqslant \sigma < 1, \\ J_k[-\pi^{-1}\log(\sigma^2-1), 0] & \text{for} \quad \sigma > 1. \end{cases} \tag{6.18}$$

Integrating (6.16) and then using (6.12) yields

$$z_k(\phi+i\psi) = \frac{2i\,e^{i\theta}}{\pi} \int_{-i}^{\phi+i\psi} (1+e^{-\pi f})^{\frac{1}{2}} \int_0^\infty \left\{ \frac{p_k(\sigma)}{1+\sigma^2+e^{-\pi f}} + \frac{q_k(\sigma)}{1-\sigma^2+e^{-\pi f}} \right\} d\sigma\, df. \tag{6.19}$$

This completes the determination of the inner expansion.

For $k = 1$, equation (6.6) yields $J_1 = \frac{1}{2}\gamma y_0$ and (6.8) yields $K_1 = \frac{1}{2}\eta''(0)x_0^2$ so $K_1' = \eta''(0)\,x_0\cos\theta$. Therefore (6.19) leads to

$$z_1(\phi+i\psi) = \frac{e^{i\theta}}{2\pi i}\left\{(4\eta''(0)\cos^3\theta + 2\gamma\cos\theta)\int_{-i}^{\phi+i\psi} \log(1+(1+e^{-\pi f})^{\frac{1}{2}})\,df + \tfrac{1}{2}\pi\gamma\,e^{i\theta}(\phi+i\psi+i)^2\right\}. \tag{6.20}$$

From (6.20) the asymptotic expansions of z_1 as $\phi \to \pm\infty$ are given by

$$z_1(\phi+i\psi) \sim \frac{-i\gamma\,e^{i2\theta}}{4}(\phi+i\psi)^2 + \frac{e^{i\theta}}{2\pi i}\{i\pi\gamma\,e^{i\theta} + \cos\theta\log 4(2\eta''(0)\cos^2\theta+\gamma)\}(\phi+i\psi) + c_1 + O(e^{-\pi(\phi+i\psi)}) \quad \text{as} \quad \phi \to +\infty, \tag{6.21}$$

$$z_1(\phi+i\psi) \sim \frac{e^{i\theta}}{2}\left\{\frac{\gamma\sin\theta}{2} + i\eta''(0)\cos^3\theta(\phi+i\psi)^2 + \frac{\gamma\,e^{i2\theta}}{2}(\phi+i\psi) + d_1 + O(e^{\frac{1}{2}\pi(\phi+i\psi)})\right\} \quad \text{as} \quad \phi \to -\infty. \tag{6.22}$$

Here c_1 and d_1 are certain specific constants whose values we shall not use.

It is to be noted that the solutions (6.10) for z_0 and (6.19) for z_k are particular ones which have the slowest rates of growth at infinity. The correctness of these choices will be verified when the inner expansion, constructed from these solutions, is matched with the outer expansions.

7. Matching the inner and outer expansions

We shall now complete the solution of the problem considered in § 6, namely the case of a wall flow which leaves the wall to become a jet. This flow is represented by three asymptotic expansions – the outer expansions for a wall flow in § 4 and for a jet flow in § 3, and the inner expansion in § 6. Each of the outer expansions contains arbitrary constants but those in the inner expansion have been chosen. We shall determine these constants by matching each of the outer expansions to the inner expansion.

Before carrying out the matching, we note that the point of detachment is arbitrary, since a flow can be found which leaves the wall at any point, provided that the resulting jet does not hit the wall immediately. This condition is met if the wall ends at the point of detachment, or if its curvature there exceeds that of the jet. It is also clear that, in the reverse flow, a jet can be found which touches the wall at any point and then flows along it. In view of this arbitrariness, we have chosen the point of detachment to be the origin for all values of ϵ. The location of the point where a real flow detaches depends upon other considerations, such as stability.

In the region $\phi < 0$ upstream from the point of detachment, the outer expansion (2.4) holds with $z_s = x_s + iy_s$ given by (4.7) and (4.8) for $s = 0$ and by (4.9) and (4.14) for $s \geqslant 1$. In the region $\phi > 0$ downstream from the point of detachment (2.4) holds with z_s given by (3.18) and (3.19) for $s = 0$ and by (3.25) and (3.29) for $s \geqslant 1$. In the neighbourhood of $\phi = 0$, expansion (2.8) holds with z_u^k given by (6.10) for $k = 0$ and by (6.19) for $k \geqslant 1$. In the definitions of ϕ'' and z'', $\phi_j(0) = 0$ is the value of ϕ at the point of detachment and $z[\phi_j(0), 0] = 0$ is the value of z at this point. Therefore from (2.5) we have

$$z(\phi + i\epsilon\psi, \epsilon) = \epsilon z''(\phi'' + i\epsilon\psi, \epsilon) \quad \text{and} \quad \phi = \epsilon\phi''.$$

Upon replacing ϕ'' by ϕ/ϵ in the argument of z'', we obtain

$$z(\phi + i\epsilon\psi, \epsilon) = \epsilon z''(\phi/\epsilon + i\psi, \epsilon). \tag{7.1}$$

By using (2.4) for z and (2.8) for z'' in (7.1), we get

$$\sum_{n=0}^{\infty} \epsilon^n z_n(\phi + i\epsilon\psi) \sim \epsilon \sum_{n=0}^{\infty} \epsilon^n z_n''(\phi/\epsilon + i\psi). \tag{7.2}$$

On the left side of (7.2) we must use the z_n for the wall flow for $\phi < 0$ and those for the jet flow for $\phi > 0$.

It is convenient to set $\phi + i\epsilon\psi = \epsilon^{\frac{3}{4}}\chi$. Then for a fixed value of χ the argument of z_n in (7.2) tends to zero as ϵ tends to zero while the argument of $z_n''(\epsilon^{-\frac{1}{4}}\chi)$ can be represented by its asymptotic expansion at infinity. Thus if we retain terms of order less than ϵ^2, we can write for the left side of (7.2)

$$z_0(\epsilon^{\frac{3}{4}}\chi) + \epsilon z_1(\epsilon^{\frac{3}{4}}\chi) + O(\epsilon^2) = z_0(0) + \epsilon^{\frac{3}{4}}\chi z_0'(0) + \tfrac{1}{2}\epsilon^{\frac{3}{2}}\chi^2 z_0''(0) + \epsilon z_1(0) + \epsilon^{\frac{7}{4}}\chi z_1'(0) + O(\epsilon^2). \tag{7.3}$$

For the right side of (7.2) we can write the upstream expansion ($\phi < 0$) by using (5.10) for z_0'' and (6.22) for z_1''. This yields

$$\epsilon z_0''(\epsilon^{-\frac{1}{4}}\chi) + \epsilon^2 z_1(\epsilon^{-\frac{1}{4}}\chi) + O(\epsilon^2) = \epsilon^{\frac{3}{4}}\chi\, e^{i\theta} + \epsilon i\, e^{i\theta} + \tfrac{1}{2}\epsilon^{\frac{3}{2}}\chi^2\, e^{i\theta} \times [\tfrac{1}{2}\gamma \sin\theta + i\eta''(0)\cos^3\theta] + \tfrac{1}{2}\epsilon^{\frac{7}{4}}\chi\gamma\, e^{i2\theta} + O(\epsilon^2), \quad \phi < 0. \tag{7.4}$$

Similarly, the downstream expansion ($\phi > 0$) of the right side of (7.2) is obtained by using (6.10) for z_0'' and (6.21) for z_1'':

$$\epsilon z_0''(\epsilon^{-\frac{1}{4}}\chi) + \epsilon^2 z_1(\epsilon^{-\frac{1}{4}}\chi) + O(\epsilon^2) = \epsilon^{\frac{3}{4}}\chi\, e^{i\theta} + \epsilon i\, e^{i\theta} - \tfrac{1}{4}\epsilon^{\frac{3}{2}}\chi^2 i\gamma\, e^{i2\theta} + \epsilon^{\frac{7}{4}}\chi \frac{e^{i\theta}}{2\pi i} \times [i\pi\gamma\, e^{i\theta} + (2\eta''(0)\cos\theta + \gamma)\cos\theta \log 4] + O(\epsilon^2), \quad \phi > 0, \tag{7.5}$$

The validity of (7.4) and (7.5) depends upon the fact that

$$z_n''(\phi''+i\psi) = O(|\phi''|^{n+1}) \quad \text{as} \quad |\phi''| \to \infty. \tag{7.6}$$

This can be proved by induction, based on a careful examination of J_k and K_k, which enter into the expression for z_k''. Then from (7.6) we have, for fixed χ,

$$\epsilon^{n+1} z_n''(\epsilon^{-\frac{1}{4}}\chi) = O(\epsilon^{\frac{3}{4}(n+1)}). \tag{7.7}$$

Thus for $n \geqslant 2$, the terms of the form (7.7) omitted from (7.4) and (7.5) are all $O(\epsilon^{\frac{9}{4}})$.

Let us now equate (7.3) for $\phi < 0$ to (7.4), which requires that the upstream values of z_0 and z_1 be used in (7.3). Equating coefficients of like powers of ϵ yields

$$\left.\begin{aligned} z_0(0) = 0, \quad z_0'(0) = e^{i\theta}, \quad z_1(0) = i\,e^{i\theta}, \\ z_0''(0) = e^{i\theta}\,[\tfrac{1}{2}\gamma\sin\theta + i\eta''(0)\cos^3\theta], \quad z_1'(0) = \tfrac{1}{2}\gamma\,e^{i2\theta}. \end{aligned}\right\} \tag{7.8}$$

From the first and third of these equations we obtain

$$x_0(0) = 0, \quad x_1(0) = -\sin\theta. \tag{7.9}$$

These two values enable us to determine uniquely the wall-flow values of $x_0(\phi)$ from (4.8) and $x_1(\phi)$ from (4.15). Then y_0 is given by (4.7) and y_1 by (4.16). The other conditions in (7.9) are then automatically satisfied, which is a check on our method and on our calculations.

Finally we must equate (7.3) for $\phi > 0$ to (7.5), using the jet-flow values of z_0 and z_1 in (7.3). Equating like powers of ϵ yields

$$\left.\begin{aligned} z_0(0) = 0, \quad z_0'(0) = e^{i\theta}, \quad z_1(0) = i\,e^{i\theta}, \quad z_0''(0) = -\tfrac{1}{2}i\gamma\,e^{i2\theta}, \\ z_1'(0) = e^{i\theta}\,[i\pi\gamma\,e^{i\theta} + \{2\eta''(0)\cos\theta+\gamma\}\cos\theta\log 4]/2\pi i. \end{aligned}\right\} \tag{7.10}$$

By using the first and second of (7.10) together with (3.17) we find that

$$a = 0, \quad \beta = \theta. \tag{7.11}$$

These conditions determine x_0 and y_0 uniquely. Similarly, the third and last of (7.10) yield $x_1(0)$, $y_1(0)$, $x_1'(0)$ and $y_1'(0)$, which suffice to determine x_1 and y_1 uniquely. The fourth condition in (7.10) is then satisfied automatically. In a similar way the constants in all the z_n with $n \geqslant 2$ can be found in both the jet and the wall flows.

We shall now summarize our results for this example. For $\phi < 0$ the wall flow is given by

$$z = x_0(\phi+i\epsilon\psi) + i\eta(x_0) + \epsilon x_1(\phi+i\epsilon\psi) + i\epsilon\left[\left\{\frac{1+[\eta'(x_0)]^2}{1-\gamma\eta(x_0)}\right\}^{\frac{1}{2}} + \eta'(x_0)\,x_1(\phi+i\epsilon\psi)\right] + O(\epsilon^2), \quad \phi < 0. \tag{7.12}$$

Here $x_0 = x_0(\phi+i\epsilon\psi)$, where $x_0(\phi)$ is the solution of (3.8) with $x_0(0) = 0$, and $x_1(\phi)$ is given by (4.15) with $\phi_0 = 0$ and $x_1(0) = -\sin\theta$. Near the point of detachment $|\phi| \ll 1$, z is given by

$$z = e^{i\theta}\,(\phi+i\epsilon\psi+i\epsilon) + \epsilon^2 z_1(\phi/\epsilon+i\psi) + O(\epsilon^3), \quad |\phi| \ll 1. \tag{7.13}$$

Here z_1 is given by (6.20).

For $\phi > 0$ the jet flow is given by

$$z = x_0(\phi+i\epsilon\psi)+i[x_0\tan\theta-\tfrac{1}{4}x_0^2\gamma\sec^2\theta]+\epsilon[\tfrac{1}{2}x_0\gamma\sec^2\theta-\tan\theta+i] \times y_1(\phi+i\epsilon\psi)+O(\epsilon^2). \quad (7.14)$$

In this case $x_0(\phi+i\epsilon\psi)$ is determined by (3.19) with $\alpha = 0$ and $\beta = \theta$ while (3.33) has been used for x_1. The initial values of x_1 and y_1 have been obtained from (7.10). Finally from (3.32), (3.34) and (3.35), y_1 is found to be

$$\begin{aligned} y_1(\phi+i\epsilon\psi) = \Bigg[\Bigg\{&\sec\theta\cos 2\theta\left(\frac{\gamma}{2}\sin 2\theta+\frac{\cos^2\theta}{\pi}\log 2[2\eta''(0)\cos\theta+\gamma]\right) \\ &-\gamma\sin 2\theta\left(\cos\theta-\frac{\sec\theta}{4}\right)\Bigg\}\left(x_0-\frac{\sin 2\theta}{\gamma}\right) \\ &+\Bigg\{2\sin\theta\left(\frac{\gamma}{2}\sin 2\theta+\frac{\cos^2\theta}{\pi}\log 2[2\eta''(0)\cos\theta+\gamma]\right) \\ &+\gamma\cos 2\theta\left[\cos\theta-\frac{\sec\theta}{4}\right]\Bigg\}\left[x_0\tan\theta-\frac{\gamma}{4}\sec^2\theta x_0^2+\frac{\cos 2\theta}{\gamma}\right]\Bigg] \\ &\times\left(1-\gamma x_0\tan\theta+\frac{\gamma^2}{4}\sec^2\theta x_0^2\right)^{-1}+\frac{\sec\theta}{4}. \end{aligned} \quad (7.15)$$

In the waterfall problem considered by Clarke (1965), $\eta(x) \equiv 0$. In that case the results (7.12)–(7.15) reduce to his.

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